Gröbner bases, monomial group actions, and the Cox rings of Del Pezzo surfaces

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Abstract

We introduce the notion of monomial group action and study some of its consequences for Gröbner basis theory. As an application we prove a conjecture of V. Batyrev and O. Popov describing the Cox rings of Del Pezzo surfaces (of degree ≥ 3) as quotients of a polynomial ring by an ideal generated by quadrics.

1 Introduction

The notion of homogeneous coordinate ring was introduced by David Cox in [1] aiming to generalize to arbitrary toric varieties the relationship between $\mathbb{P}^n$ and $k[x_0, \ldots, x_n]$. Cox's construction assigns to every toric variety T, a multigraded *polynomial ring* R (and an ideal) such that:

1. T can be recovered as a suitable quotient of $\mathrm{Spec}(R)$ by the action of a torus;
2. Modules over R correspond to sheaves on T.

This construction was generalized by Keel and Hu in [4] where the authors introduce *Cox rings*, the homogeneous coordinate rings of a much larger class of varieties. The authors show that finite generation of this ring is of fundamental importance for the birational geometry of the variety (in particular, it ensures that the Mori program can be carried out for any divisor see [4] prop. 1.11).

Moreover Keel and Hu prove that toric varieties are the only algebraic varieties whose Cox rings are polynomial rings, thus raising the question of which kinds of finitely generated k-algebras arise as Cox rings of non-toric varieties.

Probably the most important such example are the Cox rings of Del Pezzo surfaces of degree at most five. These rings were studied for the

first time by Batyrev and Popov in [6], where the authors show that they are Gorenstein k-algebras whose generators are in bijection with the (-1)-curves on the surfaces. Moreover, they conjecture that these rings are quadratic algebras.

This paper is a case study of the Cox rings of Del Pezzo surfaces, specifically of the ideals C which define them as quotients of the polynomial rings $k[E]$ (with one variable for each exceptional curve).

The groups of symmetries of the configuration of exceptional curves play a fundamental role in our study. We show that, although the action of this Weyl group on $k[E]$ does not fix the ideal C, it can be rediscovered as symmetries of the Gröbner fan of C.
This weaker form of symmetry is sufficient to characterize the monomial initial ideals of C in terms of very few values of their multigraded Hilbert Series. As an application of the techniques developed we prove Batyrev and Popov's conjecture (for surfaces of degree at least three) providing explicit generators for C.

The material is organized as follows:

- Section 2 contains background material on Del Pezzo surfaces and their Weyl groups.
- Section 3 contains the definition Cox rings and the results of Batyrev and Popov used throughout the rest of the paper.
- In Section 4 we describe the degree 2 part of the ideals C which define the Cox rings as quotients of polynomial rings.
- In Section 5 we introduce the notion of *monomial group action*, and study its consequences for Gröbner basis theory. In particular we show that, if G acts monomially on an ideal I, then it acts by symmetries on its Gröbner fan (and in particular on its tropical variety). Moreover we show that, for a general Del Pezzo surface, the corresponding Weyl group acts monomially on the ideal defining its Cox ring.
- In Section 6 we study the problem of characterizing the monomial initial ideals of a homogeneous ideal in the presence of a monomial group action compatible with the grading. As an application we characterize the monomial initial ideals of C in terms of very few values of their Picard-graded Hilbert Series.
- Section 7 contains the proof of Batyrev and Popov's conjecture (for surfaces of degree at least three). Moreover, we show that for degree al least 4 the Cox rings of Del Pezzo surfaces are Koszul algebras.

- In Section 8 we reduce the problem of finding quadratic Gröbner bases for C to a combinatorial problem about the edge ideals of the graphs of exceptional curves.
- Section 9 is an Appendix containing tables and calculations used in Section 7.

2 Del Pezzo surfaces

This section contains the required background on Del Pezzo surfaces and their Weyl groups, and introduces terminology that will be used throughout the rest of the paper.

Definition 1. *A collection of $r \leq 8$ points in $\mathbb{P}^2$ is said to be in general position if no three are on a line, no six are on a conic and any cubic containing eight points is smooth at each of them.*

Definition 2. *A Del Pezzo surface X_r is the blowup of $\mathbb{P}^2$ at $r \leq 8$ general points. The degree of X_r is $9 - r$.*

Remark 1. *Normally the definition of Del Pezzo surfaces includes $\mathbb{P}^1 \times \mathbb{P}^1$ of degree 8, but we concentrate on Del Pezzo surfaces of degree at most 5.*

Since the automorphism group of $\mathbb{P}^2$ carries any four general points to the four standard ones, there is essentially one Del Pezzo surface X_r for $r \leq 4$. In constrast, there are infinitely many nonisomorphic Del Pezzo surfaces X_r for each $r \geq 5$.

From the description of Del Pezzo surfaces as blow ups of $\mathbb{P}^2$ it follows immediately that the Picard group of X_r is isomorphic to $\mathbb{Z}^{r+1}$. A natural basis is given by:

- The pullback of the class of a line in $\mathbb{P}^2$, denoted by ℓ;
- The exceptional divisors of the blow up $e_1, \ldots, e_r$.

In terms of this basis, the intersection form is given by

$$e_i \cdot e_j = -\delta_{ij} \ , \ \ell \cdot \ell = 1 \text{ and } \ell \cdot e_j = 0$$

Moreover the canonical divisor in X_r is $K = -3\ell + e_1 + \ldots + e_r$

For $r \leq 6$, the linear system associated to the anticanonical divisor determines an embedding of X_r as a surface of degree $9 - r$ in $\mathbb{P}^{9-r}$. The best known examples of Del Pezzo surfaces are $\mathbb{P}^2$, the X_5, embedded by $|-K|$ as complete intersections of two quadrics in $\mathbb{P}^4$, and the X_6, which correspond to smooth cubic surfaces in $\mathbb{P}^3$.

Del Pezzo surfaces contain a very special collection of rational curves (for $r = 6$ these curves are the 27 lines on the cubic).

Definition 3. *An exceptional curve (also* (-1)*-curve)* C *is a curve whose class in* $\mathrm{Pic}(X_r)$ *satisfies:*

$$K \cdot C = -1 \quad \textit{and} \quad C^2 = -1$$

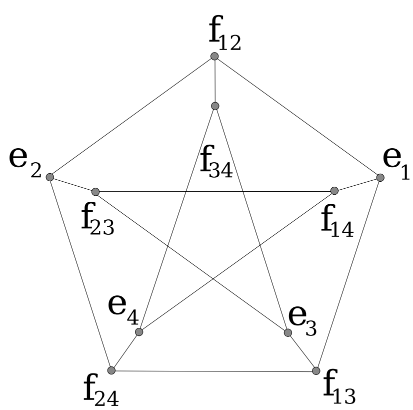
Each Del Pezzo surface contains finitely many exceptional curves which are classified (for $r \leq 6$) by the following table (see [5] for details)

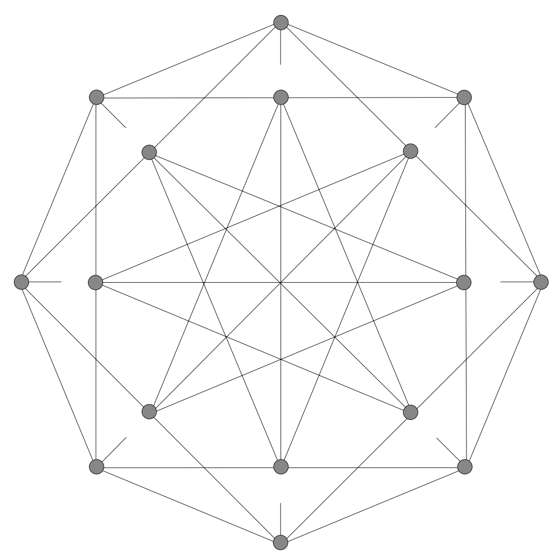
Number of blown up points	4	5	6	Class in $\mathrm{Pic}(X_r)$
Exceptional divisors e_i	4	5	6	e_i
Lines through pairs of points f_{ij}	6	10	15	$\ell - e_i - e_j$
Conics through five points g_i	0	1	6	$2\ell - \sum_{k \neq i} e_k$
Number of exceptional curves	10	16	27	

The configuration of the exceptional curves on the surface is better visualized by a graph (with multiple edges for $r \geq 7$).

Definition 4. *The graph of* (-1)*-curves is the graph with one vertex for each exceptional curve and* $E_i \cdot E_j$ *edges between edges* E_i *and* E_j *for* $i \neq j$*. We denote this graph by* L_r*.*

The graphs L_4 and L_5 (the Petersen and Clebsch graphs respectively) are shown in the figure. Note that the configuration of lines is independent of the coordinates of the blown up points.





2.1 Symmetries

For each $r \geq 2$ there is a Weyl group W_r which acts on $\mathrm{Pic}(X_r)$ by automorphisms which preserve the intersection form.

N. of Blown up points	Root system	size of W_r
4	A_4	120
5	D_5	1920
6	E_6	51840

More concretely, W_r is the subgroup of $\mathrm{Aut}(\mathrm{Pic}(X_r))$ generated by the permutations of the classes of the exceptional divisors e_i and (for $r \geq 3$) by the additional Cremona element σ given by $\sigma(\ell) = 2\ell - e_1 - e_2 - e_3$, $\sigma(e_1) = \ell - e_2 - e_3$, $\sigma(e_2) = \ell - e_1 - e_3$, $\sigma(e_3) = \ell - e_1 - e_2$ and $\sigma(e_i) = e_i$ for $i \notin \{1, 2, 3\}$.

The elements of W_r preserve the intersection form and fix the canonical divisor K. As a result they permute the classes of (-1)-curves (since they fix the equations that define them in $\mathrm{Pic}(X_r)$) and induce automorphisms of the graphs of (-1)-curves (since these permutations preserve intersection numbers). The transitivity of this action on vertices and edges explains the striking symmetry of the graphs.

3 Cox rings

The following definition was proposed by Hu and Keel in [4]

Definition 5. *Let X be a projective variety with $N^1(X) = \mathrm{Pic}(X)_{\mathbb{Q}}$ and let $L_1, \dots, L_k$ be line bundles which are a basis for the torsion free part of the Picard group and whose affine hull contains $\overline{NE}^1(X)$. A Cox ring for X is the ring*

$$CR(X, L_1, \dots, L_k) = \bigoplus_{(m_1,\dots,m_k)\in\mathbb{Z}^k} H^0\big(L_1^{\otimes m_1} \otimes \dots \otimes L_k^{\otimes m_k}\big)$$

Note that the isomorphism type of this ring is independent of the choice of basis (see [4] for details).

For a Del Pezzo surface X_r, we choose the following basis of $\mathrm{Pic}(X_r)$:

- The r line bundles $\mathcal{O}[e_i]$ corresponding to the exceptional divisors e_i;
- $\mathcal{O}[\ell]$ where ℓ is the pullback of the line $z = 0$ in $\mathbb{P}^2$.

Definition 6. *We denote by $Cox(X_r)$ the ring*

$$\begin{aligned} Cox(X_r) &= CR(X_r; \ell, e_1, \dots, e_r) = \\ &= \bigoplus_{(m_0,\dots,m_r)\in\mathbb{Z}^r} H^0(\mathcal{O}[m_0\ell + m_1e_1 + \cdots + m_re_r]) \end{aligned}$$

It is obvious from the definition that $Cox(X_r)$ is a $\mathrm{Pic}(X_r)$-graded integral domain. Moreover this ring admits a coarser $\mathbb{Z}$-grading given by

$$Cox(X_r)_n = \bigoplus_{\{D\in\mathrm{Pic}(X_r): -K\cdot D=n\}} Cox(X_r)_D \qquad \text{for } n \in \mathbb{Z}$$

Note that the above grading is nonnegative.

The Cox rings of Del Pezzo surfaces were studied for the first time by Batyrev and Popov in [6] where they show the following fundamental result:

Theorem 7. *For $3 \le r \le 7$ the ring $Cox(X_r)$ is generated by the global sections of invertible sheaves defining the exceptional curves.*

In particular, effective divisor classes can be written as sums of classes of exceptional curves.

4 Cox rings of Del Pezzo surfaces as quotients of polynomial rings

For $r < 4$ the Del Pezzo surface X_r is a toric variety and its Cox ring is a polynomial ring (see [1]). In this section we set up the notation necessary to describe $Cox(X_r)$ for $4 \le r \le 7$.

Definition 8. *Let E_r be the set of Picard classes of exceptional curves in X_r. Let $k[E_r]$ be the* $\mathrm{Pic}(X_r)$*-graded polynomial ring obtained by letting* $\deg([c]) = [c]$.

For clarity we use the symbols e_i, f_{ij}, g_i (in correspondence with the exceptional divisors, the strict transforms of lines through pairs of points and the strict transforms of conics through five points resp.) as variables in $k[E_r]$.

In this notation, Theorem 7 shows that, for every choice of nonzero global sections $s_D \in H^0(\mathcal{O}[D])$ with $D = m_0\ell + m_1e_1 + \cdots + m_re_r$ such that $[D]$ is the class of an exceptional curve, the map $\phi : k[E_r] \to Cox(X_r)$ which sends each variable to the corresponding section is a $\mathrm{Pic}(X_r)$-graded surjective homomorphism.
Note that s_D is determined by D only up to multiplication by a nonzero constant so there are many possible maps ϕ.

Definition 9. *We denote by C_r the* $\mathrm{Pic}(X_r)-$*homogeneous prime ideal $ker(\phi)$ for some choice of sections s_D. In particular $Cox(X_r) \cong k[E_r]/C_r$.*

Note that C_r depends on the choice of sections. However, multipliying the variables by constants is an automorphism of $k[E_r]$ which carries any choice of C_r to any other.

The images of monomials of $k[E_r]$ in $Cox(X_r)$ are of particular importance, we call their multiples by a nonzero constant *distinguished global sections*.

Definition 10. *A section of a bundle D is distinguished if it is supported in a union of exceptional curves on X_r.*

Note that the linear dependencies between distinguished global sections generate the ideal C_r. Describing the ideal C_r explicitly is one of the objectives of this paper, we begin by describing its (coarse) degree 2 part (as in [6]).

Definition 11. *A divisor class C on X_r is called a conic if it satisfies*

$$-K \cdot C = 2 \quad \textit{and} \quad C^2 = 0$$

It is an easy consequence of Riemann-Roch and the adjunction formula that if C is a conic then the linear system $|C|$ is base-point free and induces a morphism $X_r \to \mathbb{P}^1$ which is a conic bundle.
Moreover every such divisor has exactly $r-1$ distinguished global sections and any set of three of them are linearly dependent (3 vectors in a 2 dimensional vector space). As a result every conic provides $r-3$ linearly independent elements of C_r.

Definition 12. *We denote by $Q_r \subset C_r$ the ideal generated by the linear dependencies among distinguished global sections of conics D.*

Note that $-K \cdot D = 2$ implies that these relations are quadrics in $k[E_r]$ and is easy to see that they generate the degree 2 part of C_r (since every D with $-K \cdot D = 2$ is either a conic or contains exactly one distinguished global section).

The conic bundles C on X_r for $r \leq 6$ are described (up to permutation of the e_i's) in the following table:

r	4	5	6	Class in $\mathrm{Pic}(X_r)$
	4	5	6	$\ell - e_i$
	1	5	15	$2\ell - e_1 - e_2 - e_3 - e_4$
	0	0	6	$3\ell - 2e_1 - e_2 - e_3 - e_4 - e_5 - e_6$
	5	10	27	Total number of conics
	5	20	81	Total number of generators of Q_r

The ideals $Q_r(p_1, \ldots, p_r)$ can be generated in Macaulay2 using our package *CRDelPezzo*, which will be a part of the Macaulay2 distribution.

Example. For $r = 4$, with blown up points $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ and $[1:1:1]$ we have:

- $k[E_4] = k[f_{12}, \ldots, f_{34}, e_1, \ldots, e_4]$ graded by

$$\deg(f_{ij}) = \ell - \underline{e_i} - \underline{e_j}\ ,\ \deg(e_i) = \underline{e_i}$$

- Q_4 is the ideal generated by

Conic	Generator
$2\ell - e_1 - e_2 - e_3 - e_4$	$f_{14}f_{23} - f_{12}f_{34} - f_{13}f_{24}$,
$\ell - e_1$	$e_2f_{12} - e_3f_{13} - e_4f_{14}$,
$\ell - e_2$	$e_1f_{12} - e_3f_{23} - e_4f_{24}$,
$\ell - e_3$	$e_1f_{13} - e_2f_{23} + e_4f_{34}$,
$\ell - e_4$	$e_1f_{14} - e_2f_{24} - e_3f_{34}$

A conjecture of Batyrev and Popov. In [6] the authors conjecture that for every $4 \leq r \leq 8$ and for every Del Pezzo surface X_r the ideal C_r is generated by quadrics (i.e. $Q_r = C_r$).

Batyrev and Popov observe that the equality $Q_r = C_r$ holds up to radical. Moreover they prove that $Q_4 = C_4$ by observing that $k[E_4]/Q_4$ is the homogeneous coordinate ring of the Grassmannian $Gr(2,5)$ and hence an integral domain.

We prove Batyrev and Popov's conjecture for $r = 4, 5$ and cubic surfaces without Eckart points in Section 7. Our proof does not depend on the equality between the radicals of these ideals. It is our hope that the methods developed here could be used to characterize the ideals of relations of other Cox rings which are known to be finitely generated k-algebras.

5 Monomial group actions

Throughout the rest of the section $R = k[x_1, \ldots, x_n]$ denotes the ring of polynomials over a field k with the standard grading and G is a group acting on R by permuting the variables; $I \subset R$ is a homogeneous ideal. For the necessary background on Gröbner basis see [2].

Definition 13. *For $h \in R$, $\mathrm{mon}(h)$ is the set of monomials of R which appear with nonzero coefficient in h.*

Definition 14. *The group G acts monomially on I up to degree d if for every $h \in I$ of degree $\leq d$ and every $g \in G$ there is an element $h' \in I$ such that* $\operatorname{mon}(h') = \operatorname{mon}\big(g(h)\big)$.

Note that, if G acts monomially on I up to degree d then it acts on the set $\{\operatorname{mon}(h) : h \in I \text{ and } \deg(h) \leq d\}$.

Definition 15. *Given a monomial order $\preceq$ and an element $g \in G$, let $\preceq_g$ be the monomial order given by*

$$a \preceq_g b \Leftrightarrow g(a) \preceq g(b)$$

Lemma 16. *If G acts monomially on I up to degree d and $in_{\preceq}(I)$ is generated in degree $\leq d$ then $g(in_{\preceq}(I)) = in_{\preceq_{g^{-1}}}(I)$ for any $g \in G$.*

Proof. Let $S = \{s_1, \ldots, s_k\}$ be a reduced $\preceq$ Gröbner basis for I and let $g \in G$. By the monomiality of the action there is a set $S' = \{h_1, \ldots, h_k\} \subset I$ such that $\operatorname{mon}(h_i) = g(\operatorname{mon}(s_i))$. We show that S' is a $\preceq_{g^{-1}}$ Gröbner basis. By definition of $\preceq_{g^{-1}}$, $in_{\preceq_{g^{-1}}}(h_i) = g(in_{\preceq}(p_i))$ so

$$in_{\preceq_{g^{-1}}}(I) \supseteq \big(in_{\preceq_{g^{-1}}}(h_i)\big) \supseteq g\big((in_{\preceq}(p_i))\big) = g(in_{\preceq}(I))$$

and all these ideals coincide since the first and the last have the same Hilbert function as I (G acts on R by automorphisms of standard degree 0). As a result S' is a $\preceq_{g^{-1}}$ reduced Gröbner basis. $\square$

In particular G acts on the set of monomial initial ideals of I. We show that in fact this action extends to the Gröbner Fan of I.

Definition 17. *Given a weight vector $w = (w_1, \ldots, w_n)$ and $g \in G$, let $g(w) = (w_{g(1)}, \ldots, w_{g(n)})$.*

Definition 18. *For a weight vector w let*

$$C[w] = \{w' \in \mathbb{R}^n : \; in_{w'}(I) = in_w(I)\}$$

and let $\preceq_w$ be the monomial order defined by

$$a \preceq_w b \Leftrightarrow w \cdot a < w \cdot b \text{ or } w \cdot a = w \cdot b \text{ and } a \preceq b$$

where $\preceq$ is a fixed monomial term order and we have identified monomials with their exponent vectors.

We denote by $\preceq_{gw}$ the monomial order $(\preceq_w)_g$. Note that $\preceq_{gw}$ refines the preorder given by the weight $g(w)$.

Lemma 19. *Assume that the largest degree of a generator in any monomial initial ideal of I is d. If G acts monomially on I up to degree d then the action of G on weight vectors induces automorphisms of the Gröbner fan of the ideal I.*

Proof. Let S be a reduced $\preceq_w$ Gröbner basis and let S' be a reduced $\preceq_{g^{-1}w}$ Gröbner basis constructed as in Lemma 16.
By Proposition 2.3 in [2],

$$C[g^{-1}(w)] = \{\eta \in \mathbb{R}^n : in_\eta(s') = in_{g^{-1}w}(s') \text{ for all } s' \in S'\}$$

By definition of $g^{-1}(w)$ and S', $\eta \in C[g^{-1}(w)]$ if and only if $g(\eta)$ chooses the same leading forms of the generators of S as the weight w does. □

Similarly, under the above conditions $in_w(I)$ contains a monomial m if and only if $in_{g(w)}(I)$ contains the monomial $g^{-1}(m)$. As a result

Corollary 20. *Assume that the largest degree of a generator in any monomial initial ideal of I is d. If G acts monomially on I up to degree d then G acts by automorphisms on the tropical variety $\mathcal{T}(I)$.*

Now we show that the Weyl group W_r acts monomially on C_r for a general choice of blown up points. It should be remarked that the action of W_r on $k[E_r]$ by permutation of the coordinates does not, in general, fix the ideals C_r (even if we allow the permutations to be precomposed with diagonal matrices) so that monomiality is a way to recover the symmetries present in the configuration of exceptional curves. These reappear as symmetries of the Gröbner fan of C_r.

For clarity we denote by $C_r(p_1, \ldots, p_r)$ the ideal of relations of the Cox ring of the Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at the points $p_1, \ldots, p_r$.

Definition 21. *The group W_r acts on $(\mathbb{P}^2)^r$ by birational automorphisms. We let the symmetric group on the r indices act by permuting the coordinates and (if $r \geq 3$) we let the generator T_{123} act by*

$$\begin{array}{ccc} (\mathbb{P}^2)^r & \dashrightarrow & (\mathbb{P}^2)^r \\ (p_1, \ldots, p_r) & \longmapsto & \big(q_1, q_2, q_3, C(p_4), \ldots, C(p_r)\big) \end{array}$$

where $C : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the Cremona transformation based at p_1, p_2, p_3 and q_i is the image under C of the line between p_j and p_k, $\{i, j, k\} = \{1, 2, 3\}$.

Note that the action of W_r restricts to automorphisms of the open subset of $(\mathbb{P}^2)^r$ consisting of points $(p_1, \ldots, p_r)$ such that the blow up of $\mathbb{P}^2$ at $p_1, \ldots, p_r$ is a del Pezzo surface. We denote this open subset by U.

Let ϕ_λ be the automorphism of $k[E_r]$ obtained by multiplying each variable by a component of a vector λ of nonzero constants.

Lemma 22. *For all $g \in W_r$ there exists a λ such that*

$$\phi_\lambda(C_r(g(p_1, \ldots, p_r)) = g(C_r(p_1, \ldots, p_r))$$

In particular $m_1, \ldots, m_k$ are the monomials of an element of the ideal $C_r(p_1, \ldots, p_r)$ if and only if $g(m_1), \ldots, g(m_k)$ are those in an element of $C_r(g(p_1, \ldots, p_r))$.

Proof. Recall from Definition 9 that $C_r(p_1, \ldots, p_r)$ depends on the choice of sections s_D and that multiplying the variables by nonzero constants carries any choice of $C_r(p_1, \ldots, p_r)$ to any other. Now, linear relations between distinguished global sections are intrinsic to the surface and do not depend on its particular presentation as a blow up of $\mathbb{P}^2$. Since the blow up of $\mathbb{P}^2$ at $(p_1, \ldots, p_r)$ and at $g(p_1, \ldots, p_r)$ are two presentations of the same Del pezzo surface the result follows. ☐

Theorem 23. *For any integer d there exists an open dense set of choices for $p_1, \ldots, p_r$ such that the group W_r acts monomially on the ideal $C_r(p_1, \ldots, p_r)$ up to degree d.*

Proof. Denote by $M_{\leq d}$ the set of all subsets of monomials of $k[E_r]$ of coarse degree at most d.
For any $\underline{m} \in M_{\leq d}$ the locus $D_{\underline{m}}$ in $(\mathbb{P}^2)^r$ of points $(p_1, \ldots, p_r)$ such that the monomials in $\underline{m}$ are linearly dependent in $k[E_r]/C_r(p_1, \ldots, p_r)$ is closed: it is the locus where the rank of $\mathrm{span}(\underline{m}) \to R/C_r$ is smaller than $|\underline{m}|$. We partition the set $M_{\leq d}$ into two disjoint subsets $\mathcal{G}$ and $\mathcal{S}$, setting, for each $\underline{m} \in M_{\leq d}$,

- $\underline{m} \in \mathcal{G}$ if D_m is a proper closed subset of $(\mathbb{P}^2)^r$,
- $\underline{m} \in \mathcal{S}$ if $D_m = (\mathbb{P}^2)^r$.

The set $\mathcal{G}$ (and thus $\mathcal{S}$) is a union of orbits of G, since by Lemma 22, $D_{\underline{m}}$ does not contain $(p_1, \ldots, p_r)$ if and only if $g \cdot D_{\underline{m}}$ does not contain $g \cdot (p_1, \ldots, p_r)$.
Define

$$\mathcal{B} = \left(\mathbb{P}^2\right)^r \setminus \bigcup_{\underline{m} \in \mathcal{S}} D_{\underline{m}}$$

and observe that $\mathcal{B}$ is disjoint from $D_{\underline{m}}$, for all $\underline{m} \in \mathcal{S}$.
Let $(p_1, \ldots, p_r) \in \mathcal{B}$ and let $X = Bl_{p_1, \ldots, p_r}(\mathbb{P}^2)$. We want to prove that the action of G is monomial on $C_r(p_1, \ldots, p_r)$. We proceed by induction on the set $\mathrm{mon}(C_r) \cap M_{\leq d}$ (partially) ordered by inclusion. Suppose that $f \in C_r$, $f \neq 0$ and $\mathrm{mon}(f)$ is minimal under inclusion in $\mathrm{mon}(C_r) \cap M_{\leq d}$. In particular $D_{\mathrm{mon}(f)} \cap \mathcal{B} \neq \emptyset$ (as it contains $(p_1, \ldots, p_r)$) and therefore $\mathrm{mon}(f) \in \mathcal{G}$. Since the set $\mathcal{G}$ is W_r-invariant, for any $g \in G$ we have that $g \cdot \mathrm{mon}(f) \in \mathcal{G}$. Thus $D_{g \cdot \mathrm{mon}(f)} \supset \mathcal{B}$, and therefore there is a non-zero relation $f^g \in C_r$ such that $\mathrm{mon}(f^g) \subset g \cdot \mathrm{mon}(f)$. Note that typically $g \cdot f \notin C_r$. The

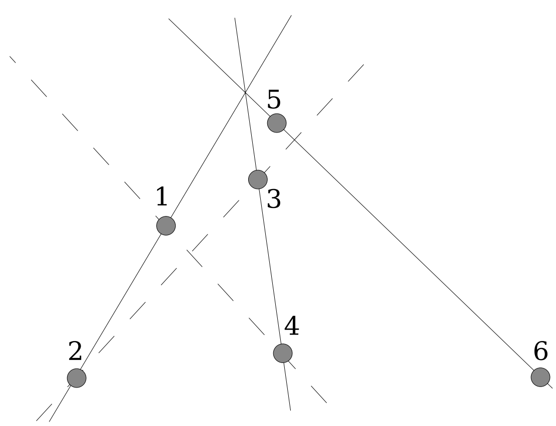
same argument applied to f^g and the group element g^{-1} implies that there is a non-zero relation $(f^g)^{g^{-1}} \in C_r$ such that $\mathrm{mon}\big((f^g)^{g^{-1}}\big) \subset g^{-1} \cdot g \cdot \mathrm{mon}(f) = \mathrm{mon}(f)$. By the minimality of $\mathrm{mon}(f)$ it follows that all the inclusions of monomials were in fact equalities and we conclude the proof of the base case of the induction.

Suppose that $f \in C_r$, $f \neq 0$ and that there is a non-zero $h \in C_r$ such that $\mathrm{mon}(h) \subsetneq \mathrm{mon}(f)$. Then we can find a constant t such that $h' := th + f$ satisfies $\mathrm{mon}(h') \subsetneq \mathrm{mon}(f)$; also by construction $0 \neq h' \in C_r$ and $\mathrm{mon}(f) = \mathrm{mon}(h) \cup \mathrm{mon}(h')$. By the induction hypothesis, for any $g \in G$ there are non-zero relations h^g and $(h')^g$ in C_r such that $\mathrm{mon}(h^g) = g \cdot \mathrm{mon}(h)$ and $\mathrm{mon}\big((h')^g\big) = g \cdot \mathrm{mon}(h')$. Denote by f^g a linear combination of h^g and $(h')^g$ where no cancellation of monomials occurs; clearly we have $\mathrm{mon}(f^g) = g \cdot \mathrm{mon}(f)$. ☐

Theorem 24. *For a general choice of points $p_1, \ldots, p_r$, the Weyl group W_r acts by symmetries on the Gröbner fan of C_r.*

Proof. Since the multigraded Hilbert Series of the ideals C_r are independent of the choice of the points, all monomial initial ideals of all C_r's have a common multigraded Hilbert function, and hence they form an antichain of monomial ideals (under inclusion). By Theorem 1.1 in [3] any such antichain is finite. Thus, the degrees of their minimal generators are uniformly bounded by an integer d. Using Lemma 19 and Theorem 23 we obtain the desired conclusion. ☐

The genericity condition in Theorem 23 is necessary: the action of the Weyl group is not monomial on special Del Pezzo surfaces. Consider a Del Pezzo X_6 with an Eckart point, that is a point q in which three exceptional curves intersect (as in the figure).



Note that the monomials $f_{12}e_1e_2, f_{34}e_3e_4$ and $f_{56}e_5e_6$ are linearly dependent in $k[E_6]/C_6(p_1, \ldots, p_6)$ since so are three lines with a common point in $\mathbb{P}^2$. On the other hand their images under the permutation

$(24) \in W_6$: $f_{14}e_1e_4$, $f_{23}e_2e_3$ and $f_{56}e_5e_6$ are linearly independent since the corresponding lines do not intersect.

6 Gröbner bases and P-graded monomial group actions

In this section the ring of polynomials R is graded by a monoid P. The grading by P refines the standard grading by degree and is refined by the grading by monomials. Recall that G acts on R by permuting the variables. Let $\preceq$ be a monomial weight order.

Definition 25. *The action of G is P-compatible if for every $g \in G$ the map $\hat{g} : P \to P$ given by $\hat{g}(D) = \deg(g(m))$ for any monomial m with $\deg(m) = D$ is a well defined endomorphism of P.*

In that case the $\hat{g}$ are in fact automorphisms of P and the action of G extends to power series via $g(\sum_{D \in P} s_D t^D) = \sum_{D \in P} a_D t^{\hat{g}(D)}$ and in particular to Hilbert Series of P-homogeneous ideals.

Lemma 26. *Let I be a P-homogeneous ideal and assume that $in_{\preceq}(I)$ is generated in degree $\leq d$. If G acts monomially on I up to degree d and the action is P-compatible then the Hilbert Series of I is G-invariant.*

Proof. By Lemma 16 we know that $g(in_{\preceq}(I)) = in_{\preceq_{g^{-1}}}(I)$, thus

$$\begin{aligned} g(HS(I)) &= g(HS(in_{\preceq}(I))) = HS(g(in_{\preceq}(I))) = HS(in_{\preceq_{g^{-1}}}(I)) = \\ &= HS(I) \end{aligned}$$

□

Note that the W_r-action on $k[E_r]$ is $\mathrm{Pic}(X_r)$ compatible.

Now, assume the action of G on R is P-compatible and also monomial on I. In the remainder of this section we discuss how, under these assumptions, partial knowledge of the Hilbert Series can be used to characterize the initial ideals of I. For a vector space V we use $|V|$ to denote its dimension.

Lemma 27. *Let $Q \subset J$ be two homogeneous ideals and let $e \in R$ be a nonzerodivisor on R/Q of multidegree $\underline{e} \in P$. Suppose that for some $D \in P$ and $a \in \mathbb{Z}_{\geq 0}$ we have $Q_{D+a\underline{e}} = J_{D+a\underline{e}}$; then $Q_D = J_D$.*

Proof. By induction on a we reduce to the case $a = 1$. Since $Q \subset J$, $Q_D \subset J_D$. Conversely, let $j \in J_D$, then $je \in J_{D+\underline{e}} = Q_{D+\underline{e}}$ so $j \in Q_D$ since e is a nonzerodivisor on R/Q. □

Lemma 28. *Let $e_1, \dots, e_m$ be variables in R with $\deg(e_i) = \underline{e}_i$. Let $Q^1, \dots, Q^m$ and $J^1, \dots, J^m$ be homogeneous ideals with $Q^i \subset J^i$ and let $s(t) = \sum_{D \in P} S_D t^D$ be a power series. If*

1. *All the J^i have the same Hilbert function, $HS(J^i, t) = s(t)$;*
2. *For each i, e_i is not a zerodivisor in R/Q^i;*
3. *All the Q^i have a common Hilbert function;*
4. *For all $D \in P$ there exists j and natural numbers a_i such that*

$$|Q^j_{D+\sum_i a_i \underline{e}_i}| = S_{D+\sum_i a_i \underline{e}_i};$$

then $Q^i = J^i$ for all i.

Proof. Let D be any multidegree in P. By (4) there are natural numbers a_i such that, for some j,

$$|Q^j_{D+\sum_i a_i \underline{e}_i}| = S_{D+\sum_i a_i \underline{e}_i}$$

Writing $\sum_i a_i \underline{e}_i = B + a_j \underline{e}_j$ with $B = \sum_{i \neq j} a_i \underline{e}_i$ and using property (1) we conclude

$$|Q^j_{D+B+a_j \underline{e}_j}| = S_{D+B+a_j \underline{e}_j} = |J^j_{D+B+a_j \underline{e}_j}|$$

Since by property (2) e_j is not a zerodivisor in R/Q^j, it follows from Lemma 27 and property (1) that

$$|Q^j_{D+B}| = |J^j_{D+B}| = S_{D+B}$$

All terms in the above equalities are independent of j so applying the same argument to the equality

$$|Q^k_{D+B}| = S_{D+B} = |J^k_{D+B}|$$

we can remove from B the $\underline{e}_k$ component; iterating this process we see that $|Q^l_D| = S_D$ for some index l (and hence for all) which proves the statement since D was arbitrary. $\square$

Lemma 29. *Let I be a homogeneous ideal in R and let G be a group acting monomially on I up to degree d and transitively on $e_1, \dots, e_m$. Let $Q \subset \mathrm{in}_{\preceq}(I)$ be a monomial ideal with generators of total degree at most d. If*

1. *The variable e_1 is not a zerodivisor in R/Q;*
2. *For all $g \in G$, $HS(R/Q, t) = HS(R/g(Q), t)$;*

3. *For all $D \in P$ there are natural numbers a_i such that*

$$|Q_{D+\sum_i a_i \underline{e}_i}| = |I_{D+\sum_i a_i \underline{e}_i}|;$$

then $Q = \mathrm{in}_{\preceq}(I)$.

Proof. Choose $g_j \in G$ such that $g_j(e_1) = e_j$. Let $Q^j = g_j(Q)$ and note that, by property (2), $HS(Q^j, t) = HS(Q, t)$ for all j.
Let $J^j = in_{\preceq_{g_j^{-1}}}(I)$; since G acts monomially on I and Q has generators of degrees at most d then $Q^j \subset J^j$. Moreover, $HS(J^j, t) = HS(I, t)$, since J^j is an initial ideal of I. Applying Lemma 28 with $s(t) = HS(I, t)$ we conclude that $Q = \mathrm{in}_{\preceq}(I)$. □

We now specialize to $R = k[E_r]$, $G = W_r$, $I = C_r(p_1, \dots, p_r)$ and assume for the rest of the section that the points $p_1, \dots, p_r$ have been chosen general enough so that W_r acts monomially on C_r up to degree d (Theorem 23).

Recall that $Cox(X_r) \cong k[E_r]/C_r$ and that the multigraded Hilbert Series of $Cox(X_r)$ is given by

$$HS(Cox(X_r), t) = \sum_{D \in \mathrm{Pic}(X_r)} h^0(\mathcal{O}_X[D]) t^D$$

so in particular it is W_r invariant (since $h^0(\mathcal{O}[D])$ depends only on intersection products which are W_r invariant).

Lemma 30. *Let $M^1 \subseteq in_{\preceq}(C_r)$ be a monomial ideal with generators of degree $\leq d$ which does not involve the variable e_1; then $M^1 = in_{\preceq}(C_r)$ if and only if the following two conditions are satisfied:*

1. *$HS(M^1, t)$ is W_r-invariant;*
2. *For $m \in \mathbb{N}$ and all $a_i \gg 0$, $\left|k[E_r]/M^1\right|_{m\ell + \sum_i a_i e_i} = \binom{m+2}{2}$.*

Proof. Apply Lemma 29 with $Q = M^1$. Hypotheses (1) and (2) of Lemma 29 follow immediately from our assumptions thus we verify hypothesis (3). For that let D be any divisor and note that there exist natural numbers $a_i \gg 0$ such that

$$D + \sum a_i e_i = m\ell + \sum b_i e_i$$

with $b_i \geq 0$. Thus $h^0(\mathcal{O}[D + \sum a_i e_i]) = h^0(\mathcal{O}[m\ell]) = \binom{m+2}{2}$ and by 2 above we have

$$\left|k[E_r]/M^1\right|_{D+\sum a_i e_i} = \binom{m+2}{2} = |Cox(X_r)|_{D+\sum a_i e_i}$$

Thus, for any $D \in \mathrm{Pic}(X_r)$ there are natural numbers a_i such that

$$|M^1|_{D+\sum a_i e_i} = |C_r|_{D+\sum a_i e_i}$$

as we wanted to prove. If conversely $M^1 = in_{\preceq}(C_r)$, 1 and 2 are obvious from the remark preceding this Lemma. $\square$

The third condition in the last Theorem is easy to verify.

Lemma 31. *Let M be a monomial ideal in $k[E_r]$ and let $D = m\ell + a_1e_1 + \cdots + a_re_r \in P$. If for $m \in \mathbb{N}$ and $a_i \gg 0$ the dimensions $\left|k[E_r]/M\right|_D$ or $\left|k[E_r]/(M : (e_1 \dots e_r)^\infty)\right|_D$ do not depend on the a_i's, then*

$$\left|k[E_r]/M\right|_D = \left|k[E_r]/(M : (e_1 \dots e_r)^\infty)\right|_D$$

Proof. Let $u = e_1e_2 \dots e_r$ and let k be such that $(M : u^k) = (M : u^\infty)$. Consider the exact sequence

$$0 \to k[E_r]/(M : u^k)[-\deg(u^k)] \to k[E_r]/M \to k[E_r]/(M + (u^k)) \to 0$$

and note that $\left|k[E_r]/(M + (u^k))\right|_D = 0$ for all m and all a_i sufficiently large. As a result the first map determines an isomorphism between the corresponding graded components. $\square$

7 The homogeneous coordinate rings for Del Pezzo surfaces

As an application of the techniques developed so far, we prove in this section that for $r = 4, 5$ and cubic surfaces without Eckart points the ideal of relations $C_r(p_1, \dots, p_r)$ of the Cox ring of every Del Pezzo surface X_r is generated by the quadrics $Q_r(p_1, \dots, p_r)$ coming from conic divisor classes.
As remarked earlier this fact was conjectured by Batyrev and Popov in [6].

Theorem 32. *Assume that $M \subset in_{\preceq}(Q_r(p_1, \dots, p_r))$ is a monomial ideal which does not involve all variables. If $p_1, \dots, p_r$ are sufficiently general then the following are equivalent:*

1. *$M = in_{\preceq}(Q_r)$ and $Q_r = C_r$;*
2. *(a) $HS(M, t)$ is W_r-invariant,*
   (b) For $m \in \mathbb{N}$ and all $a_i \gg 0$, $\left|k[E_r]/M^1\right|_{m\ell+\sum_i a_ie_i} = \binom{m+2}{2}$.

Proof. Clearly 1 implies 2 since these properties are obviously satisfied by the Hilbert Series of the Cox ring. If 2 holds, note that

$$M \subseteq in_{\preceq}(Q_r) \subseteq in_{\preceq}(C_r)$$

so that by Lemma 30, $M = in_{\preceq}(C_r)$. Thus Q_r and C_r have a common initial ideal and in particular the same Hilbert Series. Since $Q_r \subseteq C_r$ the ideals coincide. Note that the genericity of $p_1, \ldots, p_r$ was used to guarantee that W_r acted monomially on C_r up to degree d, a necessary condition for Lemma 30. $\square$

Let $S \subset (\mathbb{P}^2)^r$ be the open set of r-tuples of points in general position (no three on a line, no six on a conic) and let $U \subset S$ be open and nonempty.

Theorem 33. *Assume that M_r is a monomial ideal which does not involve all variables. If*

1. *$M_r \subset in_{\preceq}(Q_r(p_1, \ldots, p_r))$ for all $(p_1, \ldots, p_r) \in U$;*
2. *$HS(M_r, t)$ is W_r-invariant;*
3. *For $m \in \mathbb{N}$ and all $a_i \gg 0$, $\left|k[E_r]/M^1\right|_{m\ell + \sum_i a_i e_i} = \binom{m+2}{2}$;*

then $C_r = Q_r$ for all Del Pezzo surfaces X_r obtained by blowing up $\mathbb{P}^2$ at points $(p_1, \ldots, p_r) \in U$. Moreover M_r is a common initial ideal of all $C_r(p_1, \ldots, p_r)$ for $(p_1, \ldots, p_r) \in U$.

Proof. Since E is irreducible, U and the open set on which the action W_r is monomial guaranteed by Theorem 23 must intersect. Thus, for a sufficiently general choice of points $p_1, \ldots, p_r$, Theorem 32 implies that $M_r = in_{\preceq}(C_r(p_1, \ldots, p_r))$. Since the Multigraded Hilbert Series of the Cox ring is independent of the coordinates of the points we conclude that $M_r = in_{\preceq}(C_r(p_1, \ldots, p_r))$ for *every* Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at points in U.
On the other hand the inclusions

$$Q_r \subseteq C_r \quad \text{and} \quad in_{\preceq}(C_r) = M_r \subseteq in_{\preceq}(Q_r)$$

imply that the initial ideals (and hence the Hilbert Series) of C_r and Q_r coincide for all Del Pezzo surfaces X_r for $(p_1, \ldots, p_r) \in U$ and hence that $C_r = Q_r$ for all $(p_1, \ldots, p_r) \in U$. $\square$

We now construct the monomial ideals M_r.

7.1 The Del Pezzo surface X_4

In this section $\preceq$ denote the reverse lexicographic on $k[E_4]$ with

$$e_1 \succ e_2 \succ e_3 \succ e_4 \succ f_{12} \succ f_{13} \succ f_{23} \succ f_{14} \succ f_{24} \succ f_{34}$$

For a finite subset $A \subset \mathrm{Pic}(X_r)$ and $d \in \mathrm{Pic}(X_r)$ let $A + d = \{a + d : a \in A\}$, $2A = \{a + a : a \in A\}$ and $t^A = \sum_{a \in A} t^a$.

Lemma 34. *The ideal $M_4 = (f_{23}f_{14}, e_1f_{14}, e_1f_{13}, e_2f_{12}, e_1f_{12})$ has the following properties:*

1. *The generators of M_4 do not involve e_4;*
2. *$M_4 \subseteq in_{\preceq}(Q_4)$;*
3. *The Multigraded Hilbert Series of $k[E_4]/M_4$ is W_4-invariant;*
4. *For $m \in \mathbb{N}$ and all $a_i \gg 0$, $|k[E_r]/M_4|_{m\ell+\sum_i a_i e_i} = \binom{m+2}{2}$.*

Proof. (1.) and (2.) are obvious (see generating set of Q_4 in Section 4). Direct computation shows that the Hilbert Series of $k[E_4]/M_4$ is given by

$$1 - t^C + t^{-K-C} - t^{-K}$$

where C denotes the set of conic bundles in X_4. Hence (3.) follows since each of the summands is W_4-invariant.
Finally, Lemma 31 shows that it suffices to verify (4.) on

$$\begin{aligned} k[E_4]/(M_4 : e_1 \dots e_4) &= k[E_4]/(f_{12}, f_{13}, f_{14}) = \\ &= k[e_1, e_2, e_3, e_4, f_{23}, f_{24}, f_{34}] \end{aligned}$$

In this ring, the monomials $e_1^{s_1}e_2^{s_2}e_3^{s_3}e_4^{s_4}f_{23}^{h_{23}}f_{24}^{h_{24}}f_{34}^{h_{34}}$ of multidegree $m\ell + a_1\underline{e}_1 + \cdots + a_4\underline{e}_4$ with $m, a_i \geq 0$ correspond to (non-negative) integral solutions of the system of equations:

$$\begin{aligned} s_1 &= a_1 \\ s_2 - h_{23} - h_{24} &= a_2 \\ s_3 - h_{23} - h_{34} &= a_3 \\ s_4 - h_{24} - h_{34} &= a_4 \\ h_{23} + h_{24} + h_{34} &= m \end{aligned}$$

There are $\binom{m+2}{2}$ such solutions (for all $a_i \gg 0$) since they are determined by decompositions of m as a sum of three non-negative integers. $\square$

Corollary 35. *$Cox(X_4) \cong k[E_4]/Q_4$.*

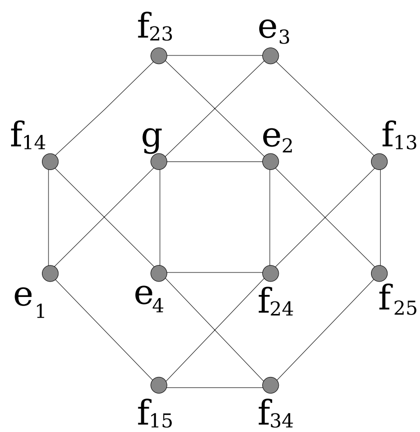
Proof. Follows immediately from Lemma 34 and Theorem 33. $\square$

7.2 The Del Pezzo surfaces X_5

Let $\preceq$ be any monomial order on $k[E_5]$ refining the one determined by the following weights:

$$\begin{array}{cccccccc} e_4 & f_{13} & e_2 & f_{23} & f_{15} & g & e_1 & f_{25} \\ 13 & 11 & 10 & 9 & 8 & 8 & 7 & 7 \\ \\ e_3 & e_5 & f_{12} & f_{14} & f_{24} & f_{34} & f_{35} & f_{45} \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 1 \end{array} \tag{1}$$

and let M_5 denote the edge ideal of the following graph (the ideal generated by products of variables which are connected by an edge).



Lemma 36. *The ideal M_5 is contained in the $\preceq$ initial ideal of Q_5 for every Del Pezzo surface X_5.*

Proof. On a Del Pezzo surface any three sections of a conic bundle D are linearly dependent. As a result, given any three monomials of degree D and any choice of points $p_1, \ldots, p_5$ some linear combination of the monomials lies in $Q_5(p_1, \ldots, p_5)$. Moreover the coefficients of this linear combination must be all nonzero since any two distinct distinguished global sections (i.e. monomials) of D have different support and are therefore independent.

As a result, to show that a given monomial m of degree D is a $\preceq$-initial term of $Q_5(p_1, \ldots, p_5)$ for all Del Pezzo surfaces X_5 it suffices to find two more monomials of degree D which are $\preceq$-smaller.

The $\preceq$ weights of all monomials in each conic D are recorded in the

table.

$\mathrm{Pic}(X)$ – degree	Monomials and their weights							
$\ell - e_1$	e_4f_{14}	19	e_3f_{13}	17	e_2f_{12}	16	e_5f_{15}	14
$\ell - e_2$	e_4f_{24}	19	e_3f_{23}	15	e_1f_{12}	13	e_5f_{25}	13
$\ell - e_3$	e_4f_{34}	19	e_2f_{23}	19	e_1f_{13}	18	e_5f_{35}	12
$\ell - e_4$	e_2f_{24}	16	e_1f_{14}	13	e_3f_{34}	12	e_5f_{45}	7
$\ell - e_5$	e_2f_{25}	17	e_1f_{15}	15	e_4f_{45}	14	e_3f_{35}	12
$2\ell - e_1 - e_2 - e_3 - e_4$	$f_{24}f_{13}$	17	$f_{14}f_{23}$	15	e_5g	14	$f_{12}f_{34}$	12
$2\ell - e_1 - e_2 - e_3 - e_5$	e_4g	21	$f_{13}f_{25}$	18	$f_{23}f_{15}$	17	$f_{12}f_{35}$	12
$2\ell - e_1 - e_2 - e_4 - e_5$	e_3g	14	$f_{24}f_{15}$	14	$f_{14}f_{25}$	13	$f_{12}f_{45}$	7
$2\ell - e_1 - e_3 - e_4 - e_5$	e_2g	18	$f_{15}f_{34}$	14	$f_{13}f_{45}$	12	$f_{14}f_{35}$	12
$2\ell - e_2 - e_3 - e_4 - e_5$	e_1g	15	$f_{25}f_{34}$	13	$f_{24}f_{35}$	12	$f_{23}f_{45}$	10

Note that the generators of M_5 are the first two monomials in each row. They have higher weights than the last two and hence $M_5 \subset in_{\preceq}(Q_5(p_1, \ldots, p_5))$ for all Del Pezzo surfaces X_5. □

Lemma 37. *The ideal M_5 has the following properties:*

1. *The generators of M_5 do not involve the variable e_5;*
2. *$HS(M_5, t)$ is W_5-invariant;*
3. *For $m \in \mathbb{N}$ and all $a_i \gg 0$, $\left|k[E_5]/M_5\right|_{m\ell + \sum_i a_i e_i} = \binom{m+2}{2}$.*

Proof. To verify (2.) we describe the multigraded Hilbert Series of $k[E_5]/M_5$ explicitly. Consider the following sets of divisors on X_5 (note that each set is invariant under the action of W_5).

Set	Divisors
C	Conic bundles
D	Divisors $d = c + v$ for $c \in C$, $v \in E_5$ with $c \cdot v = 1$
F	Divisors $f = 2c + v$ for $c \in C$, $v \in E_5$ with $c \cdot v = 1$
G	Divisors $g = c_1 + c_2$ for $c_i \in C$, $c_1 \cdot c_2 = 2$
H	Divisors $h = 2c_1 + c_2$ for $c_i \in C$, $c_1 \cdot c_2 = 2$
J	Divisors $j = c_1 + c_2 + c_3$ for $c_i \in C$, $c_i \cdot c_j = 2$ for $i \neq j$
K	Canonical divisor

Now let

$$\alpha = 1 - 2t^C + 3t^D + t^{2C} - t^F - 3t^{-K} - 6t^G$$

Direct computer computation shows that the numerator of the Hilbert Series of $k[E_5]/M_5$ is given by:

$$(\alpha + \alpha^*) + t^J + 12t^H \tag{2}$$

where $(-)^*$ is linear and $(t^A)^*$ denotes t^{-3K-A}.
It follows that the Hilbert Series is invariant under the action of W_5 since each term of (2) is.
By Lemma 31 we can verify (3.) in the quotient

$$\begin{gathered} k[E_5]/\big(M_5 : (e_1 \dots e_5)^\infty\big) = \\ k[E_5]/\big(f_{13}, f_{23}, f_{14}, f_{24}, f_{34}, f_{15}, f_{25}, g\big) = \\ = k[e_1, e_2, e_3, e_4, e_5, f_{12}, f_{35}, f_{45}] \end{gathered}$$

The monomials $e_1^{s_1} e_2^{s_2} e_3^{s_3} e_4^{s_4} e_5^{s_5} f_{12}^{h_{12}} f_{35}^{h_{35}} f_{45}^{h_{45}}$ in this ring of multidegree $m\ell + a_1 e_1 + \cdots + a_5 e_5$ with $m, a_i \geq 0$ correspond to non-negative integral solutions of the system of equations:

$$\begin{array}{rcl} s_1 - h_{12} & = & a_1 \\ s_2 - h_{12} & = & a_2 \\ s_3 - h_{35} & = & a_3 \\ s_4 - h_{45} & = & a_4 \\ s_5 - h_{35} - h_{45} & = & a_5 \\ h_{12} + h_{35} + h_{45} & = & m \end{array}$$

For all sufficiently large a_i these solutions correspond to ways of writing a as a sum of three natural numbers: there are $\binom{m+2}{2}$ such monomials. □

Corollary 38. *For all Del Pezzo surfaces X_5 of degree 4 we have $C_5(p_1, \dots, p_5) = Q_5(p_1, \dots, p_5)$. In particular*

$$Cox\big(X_5(p_1, \dots, p_5)\big) \cong k[E_5]/Q_5(p_1, \dots, p_5)$$

Corollary 39. *In the notation of Lemma 37 the multigraded Hilbert Series of $Cox(X_5)$ is given by*

$$HS(Cox(X_5)) = \alpha + t^J + 12t^H + \alpha^*$$

Corollary 40. *The ideal M_5 is an initial ideal of $C_5(p_1, \dots, p_5)$ for all Del Pezzo surfaces X_5. In particular the rings $Cox(X_r)$ for $r \leq 5$ are Koszul algebras.*

Proof. The corollaries follow immediately from Lemmas 36 and 37 and Theorem 33. The last statement follows from the well known fact that every G-quadratic algebra is Koszul. □

7.3 The Del Pezzo surfaces X_6

Recall that every nef divisor class D on a Del Pezzo surface can be written as

$$D = n_8(-K_{X_8}) + \dots + n_2(-K_{X_2}) + D'$$

where $X_8 \to X_7 \to \ldots \to X_1$ is a sequence of contractions of $(-1)-$curves, K_{X_i} is the canonical divisor of X_i, the $n_i \geq 0$ and D' is a nef divisor on X_1, a surface which is either the blow up of $\mathbb{P}^2$ at one point or $\mathbb{P}^1 \times \mathbb{P}^1$.
Using this fact we show that, for every Del Pezzo surface X_6, the ideals Q_6 and C_6 coincide in those degrees D with $D^2 = 1$ and $-K \cdot D = 3$.

Lemma 41. *Let X_6 be a Del Pezzo surface of degree 3 and let $D \in \operatorname{Pic}(X_6)$ be a nef divisor such that $D^2 = 1$ and $-K \cdot D = 3$. Then there is a morphism $\pi : X_6 \to \mathbb{P}^2$ exhibiting X_6 as the blow up of $\mathbb{P}^2$ at 6 points such that $D = \pi^* \ell$, where ℓ is the divisor class of a line in $\mathbb{P}^2$.*

Proof. Write

$$D = n_6(-K_{X_6}) + \ldots + n_2(-K_{X_2}) + D'$$

with $n_j \geq 0$ and D' a nef divisor on either the blow up of $\mathbb{P}^2$ at one point or on $\mathbb{P}^1 \times \mathbb{P}^1$. Since all the divisors appearing on the right are nef, all intersection numbers among them are non-negative. In particular the condition $D^2 = 1$ implies $n_6 = \ldots = n_2 = 0$.
Now, $\mathbb{P}^1 \times \mathbb{P}^1$ admits no divisor of square 1 (its Picard group is generated by the rulings h_1, h_2 with $h_1^2 = h_2^2 = 0$ and $h_i h_j = 1$) and the blow up of $\mathbb{P}^2$ at one point has $D = \ell$ as only solution of the equations $D^2 = 1$ and $-K \cdot D = 3$. Thus D is the pull-back of the divisor class of a line under a birational morphism $X \to \mathbb{P}^2$. □

Lemma 42. *Let X be a Del Pezzo surface of degree at most 6. The ideals Q_6 and C_6 coincide in all degrees $D \in P$, where D is a nef divisor satisfying $D^2 = 1$ and $K \cdot D = -3$.*

Proof. The Riemann-Roch formula and the Kodaira vanishing Theorem imply that $\dim Cox(X)_D = 3$. Since $k[E_r]/Q_r \to k[E_r]/C_r = Cox(X)$ is surjective, the result follows if we show $\dim\big(k[E_r]/Q_r\big)_D \leq 3$.

Lemma 41 implies that we can find a morphism $\pi : X \to \mathbb{P}^2$ which is the blow up of m points $p_1, \ldots, p_m$, such that D is the pull-back of the divisor class of a line in $\mathbb{P}^2$. Denote by $\underline{g}_1, \ldots, \underline{g}_m$ be the divisor classes of the exceptional divisors of π and denote by g_i the variable of $k[E_r]$ corresponding to $\underline{g}_i$ and by h_{ij} the variable corresponding to the strict transform of the line through the two blown up points p_i and p_j. Any element $f \in k[E_r]$ of degree D is a linear combination of monomials $h_{ij} g_i g_j$, with $1 \leq i < j \leq m$. It is enough to show that f is a linear combination of $h_{ij} g_i g_j$, with $1 \leq i < j \leq 3$, modulo Q_r.

Since Q_r and C_r agree up to coarse degree 2, the ideal Q_r contains relations of the form $q = q_{ij} h_{ij} g_j + q_{ri} h_{ri} g_r + q_{si} h_{si} g_s$, where q_{ij}, q_{ri}, q_{si} are nonzero constants. Thus if $i < j$ and $j \geq 4$, we can choose distinct

$r, s \in \{1, 2, 3\} \setminus \{i\}$ and use the relation qg_i and induction on j to achieve our goal. $\square$

Let $\preceq$ be the monomial order obtained by refining the following weight vector with the reverse lexicographic order:

3	3	3	2	2	2	2	2	2
f_{23}	g_3	e_2	g_2	g_6	e_6	f_{26}	f_{24}	g_4
2	2	2	2	2	2	2	2	2
f_{45}	e_5	f_{34}	f_16	e_4	f_{14}	f_{12}	g_1	f_{46}
2	2	2	2	2	2	1	1	1
e_3	f_{36}	f_{35}	f_{13}	f_{56}	f_{25}	e_1	g_5	f_{15}

We now construct a monomial ideal M_6 which is a $\preceq$-initial ideal of C_r for every cubic surface without Eckart points. The tables mentioned in its description are contained in the Appendix (Section 9).
Let M_6 be the monomial ideal generated by the 81 quadratic monomials in the first three columns of Table T1, the 34 cubic monomials in the first column of Table T2 and the first cubic monomial in Table T3. Note that the monomials of each Picard degree D have been written in decreasing $\preceq$-order in the Tables.

Recall that an Eckart point on a Del Pezzo surface X_6 is a point in which 3 exceptional curves intersect.

Lemma 43. *The inclusion $M_6 \subset in(Q_6(p_1, \ldots, p_6))$ holds for every Del Pezzo surface X_6 without Eckart points.*

Proof. The quadratic generators of M_6 are initial terms of Q_6 for every Del Pezzo surface X_6 (appearing in the relation which involves them and the two last monomials in each row of Table T1 in the Appendix). For the cubic generators of M_6 there are two cases depending on their multidegree D.
If $D^2 = 1$ then $h^0(O[D]) = 3$ so every four monomials in D are linearly dependent. Moreover, all coefficients of any linear dependence relation are nonzero since if X_6 has no Eckart points, Lemma 41 implies that every three monomials in D are linearly independent. In particular (by Lemma 42) the first term in each row of Table T2 is an element of $in_{\preceq}(Q_6(p_1, \ldots, p_6))$.
Finally, it follows from Lemma 51 (in the Appendix) that the monomial $f_{46}f_{13}f_{25}$, the only cubic generator of M_6 of degree $-K$, lies in $in_{\preceq}(Q_6)$ for every cubic surface without Eckart points. $\square$

Note that a general Del Pezzo surface X_6 has no Eckart points.

Direct calculations using Macaulay2 show that:

Lemma 44. *The ideal M_6 has the following properties:*

1. *The generators of M_6 do not involve the variable g_5;*
2. *$HS(M_6, t)$ is W_6-invariant;*
3. *For $m \in \mathbb{N}$ and all $a_i \gg 0$, $|k[E_6]/M_6|_{m\ell+\sum_i a_i e_i} = \binom{m+2}{2}$.*

Corollary 45. *For all Del Pezzo surfaces X_6 without Eckart points, $C_6 = Q_6$. In particular*

$$Cox\big(X_6(p_1, \ldots, p_5)\big) \cong k[E_6]/Q_6(p_1, \ldots, p_6)$$

Proof. Follows from Lemmas 43 and 44 and Theorem 33 where U is the set of points $(p_1, \ldots, p_r)$ in general position such that the corresponding blow up of $\mathbb{P}^2$ is a Del Pezzo surface without Eckart points. $\square$

8 Quadratic Gröbner basis

For $r = 4$ and 5, the quadratic initial ideals which we have exhibited are edge ideals of subgraphs of the graphs of exceptional curves L_r. As we will show, this is no accident.

Note that the edges of L_r can be colored by the conic bundles by assigning $\deg(v_1 v_2)$ to the edge $\{v_1, v_2\}$. Each color class contains exactly $r - 1$ edges.

Lemma 46. *If M is a quadratic initial ideal of C_r then M is the edge ideal of a subgraph of the graph of exceptional curves with $r - 3$ edges on each color class.*

Proof. If $D = \ell - e_1$, a basis for R_D is given by the $r - 1$ monomials $f_{1i} e_i$, for $i = 2, \ldots, r$. Since W_r acts transitively on conic bundles D we see that $|R_D| = r - 1$, and that the monomials in this graded component correspond to pairs of (-1)-curves which intersect, that is, to edges of the graph H_r on the conic bundle D.
Now, $|R/M|_D = |R/C_r|_D = h^0(\mathcal{O}[D]) = 2$ so $|M|_D = r - 3$ for all conic bundles. If $-K \cdot D = 2$ and D is effective but not a conic bundle then $D = C_1 + C_2$ and the curves C_1, C_2 either do not intersect or are equal. As a result $C_j \cdot D < 0$ and R_D is spanned by the monomial $c_1 c_2$. Since $h^0(\mathcal{O}[D]) = 1$, $|M_D| = 0$ and the statement follows. $\square$

In view of Lemma 46 and Theorem 33 it becomes a question of interest to characterize all subgraphs of the graphs of exceptional curves whose multigraded Hilbert Series is W_r-invariant. For $r \leq 5$ this can be accomplished by direct computer exploration and one can show,

Lemma 47. *There are exactly 18 quadratic initial ideals of C_5 up to the action of $W_5 = D_5$.*

The corresponding weight vectors are in Table T4 in the Appendix.

For $r = 6$ the space of possible subgraphs is much larger and exhaustive exploration is simply unfeasible. One can in fact show that there are no "small" quadratic initial ideals, that is, initial ideals whose generators involve less than 25 variables. This observation depends on the geometry of the 27 lines on the cubic and we will prove it.

Lemma 48. *Let $A = \{a_1, a_2, a_3\}$ be a set of three distinct exceptional curves. Then the exceptional curves in A do not form a triangle if and only if there exist sets of (-1)-curves $C = \{c_1, \ldots, c_6\}$ and $H = \{h_2, \ldots, h_6\}$ disjoint from A such that*

1. *The h_i are pairwise disjoint;*
2. *h_i intersects c_i and c_1 and no other curve in C;*
3. *There is a conic bundle D whose distinguished global sections are precisely the c_ih_i.*

Proof. If the curves in A form a triangle, it is easy to see that every conic bundle D contains exactly one monomial divisible by a variable in A so (3.) is impossible.
Conversely note that $C = \{e_1, \ldots, e_6\}$ and $H = \{f_{12}, f_{13}, \ldots f_{16}\}$ satisfy conditions (1.), (2.) and (3.) (with $D = \ell - e_1$). Moreover, the sets of curves $\{f_{23}, f_{24}, f_{25}\}$, $\{f_{23}, f_{46}, f_{45}\}$ and $\{f_{23}, f_{46}, g_1\}$ induce subgraphs with all isomorphism types of graphs of size 3 except the triangle. If the curves in A form any such graph, the action of the Weyl group can carry them into one of these. As a result their complement contains the required sets of exceptional curves. $\square$

Theorem 49. *If there is a quadratic initial ideal N for Q_6, its generators must involve at least 25 of the 27 variables.*

Proof. If N involves 24 variables or less pick three $\{v_1, v_2, v_3\}$ which do not appear in the generators and denote by S the subgraph of G_6 that they span.
If this subgraph is not a triangle, Lemma 48 shows that there are sets of curves $C = \{c_1, \ldots, c_6\}$ and $H = \{h_1, \ldots, h_5\}$ disjoint from S such that h_i intersects c_i and c_1 and no other curve in C and the c_ih_i are all distinguished sections of the conic bundle D.
By Lemma 46 two distinguished sections of D, say c_1h_1 and c_2h_2 are not edges of the subgraph of L_6 corresponding to N. As a result

$$c_1\,,\, c_2\,,\, c_3\,,\, c_4\,,\, c_5\,,\, h_1\,,\, h_2\,,\, v_1\,,\, v_2\,,\, v_3$$

is a set of independent vertices (i.e. no two joined by an edge in G_N) so $\mathrm{codim}(N) \leq 17 < \mathrm{codim}(Q_6)$ and N cannot be an initial ideal.
If the subgraph S is a triangle, then the situation is very different

and there are subgraphs of G_6 with the same Hilbert function as Q_6. Computer calculations done with Macaulay2 show that they are not initial ideals for any weight vector. □

9 Appendix: Generators of M_6

This section contains explicit calculations that were used in the proof of Lemma 43. All polynomials have been written in decreasing $\preceq$-order. M_6 is generated by the 81 quadratic monomials in the first two columns of Table T1, the 34 cubic monomials in the first column of Table T2 and the first monomial in Table T3.

T1					
$2\ell - e_1 - e_2 - e_3 - e_5$	e_6g_4	g_6e_4	$f_{12}f_{35}$	$f_{13}f_{25}$	$f_{23}f_{15}$
$3\ell - e_1 - 2e_2 - e_3 - e_4 - e_5 - e_6$	$f_{23}g_3$	g_6f_{26}	$f_{24}g_4$	$f_{12}g_1$	$f_{25}g_5$
$2\ell - e_1 - e_2 - e_3 - e_6$	$f_{23}f_{16}$	g_4e_5	$f_{12}f_{36}$	$f_{26}f_{13}$	e_4g_5
$2\ell - e_1 - e_2 - e_4 - e_5$	g_3e_6	$f_{45}f_{12}$	g_6e_3	$f_{14}f_{25}$	$f_{24}f_{15}$
$3\ell - e_1 - e_2 - 2e_3 - e_4 - e_5 - e_6$	$f_{23}g_2$	g_4f_{34}	g_6f_{36}	g_1f_{13}	$f_{35}g_5$
$\ell - e_1$	e_2f_{12}	e_6f_{16}	e_4f_{14}	e_3f_{13}	e_5f_{15}
$2\ell - e_1 - e_2 - e_4 - e_6$	g_3e_5	$f_{24}f_{16}$	$f_{26}f_{14}$	$f_{12}f_{46}$	e_3g_5
$2\ell - e_1 - e_3 - e_4 - e_5$	e_2g_6	g_2e_6	$f_{14}f_{35}$	$f_{45}f_{13}$	$f_{34}f_{15}$
$3\ell - e_1 - e_2 - e_3 - 2e_4 - e_5 - e_6$	g_3f_{34}	g_2f_{24}	$f_{14}g_1$	g_6f_{46}	$f_{45}g_5$
$\ell - e_2$	$f_{23}e_3$	e_6f_{26}	$f_{24}e_4$	e_5f_{25}	$f_{12}e_1$
$2\ell - e_1 - e_2 - e_5 - e_6$	g_3e_4	g_4e_3	$f_{12}f_{56}$	$f_{16}f_{25}$	$f_{26}f_{15}$
$2\ell - e_1 - e_3 - e_4 - e_6$	g_2e_5	$f_{34}f_{16}$	$f_{14}f_{36}$	$f_{46}f_{13}$	e_2g_5
$2\ell - e_2 - e_3 - e_4 - e_5$	$f_{23}f_{45}$	e_6g_1	$f_{24}f_{35}$	$f_{34}f_{25}$	g_6e_1
$3\ell - e_1 - e_2 - e_3 - e_4 - 2e_5 - e_6$	g_3f_{35}	g_4f_{45}	g_6f_{56}	g_2f_{25}	g_1f_{15}
$\ell - e_3$	$f_{23}e_2$	$f_{34}e_4$	e_6f_{36}	e_5f_{35}	$f_{13}e_1$
$2\ell - e_1 - e_3 - e_5 - e_6$	e_2g_4	g_2e_4	$f_{16}f_{35}$	$f_{13}f_{56}$	$f_{36}f_{15}$
$2\ell - e_2 - e_3 - e_4 - e_6$	$f_{23}f_{46}$	$f_{26}f_{34}$	e_5g_1	$f_{24}f_{36}$	e_1g_5
$3\ell - e_1 - e_2 - e_3 - e_4 - e_5 - 2e_6$	g_3f_{36}	g_2f_{26}	$f_{16}g_1$	g_4f_{46}	$f_{56}g_5$
$\ell - e_4$	e_2f_{24}	$f_{45}e_5$	e_6f_{46}	$f_{34}e_3$	$f_{14}e_1$
$2\ell - e_2 - e_3 - e_5 - e_6$	$f_{23}f_{56}$	e_4g_1	$f_{26}f_{35}$	$f_{36}f_{25}$	g_4e_1
$2\ell - e_1 - e_4 - e_5 - e_6$	g_3e_2	$f_{45}f_{16}$	g_2e_3	$f_{14}f_{56}$	$f_{46}f_{15}$
$\ell - e_5$	e_2f_{25}	$f_{45}e_4$	e_3f_{35}	e_6f_{56}	e_1f_{15}
$2\ell - e_2 - e_4 - e_5 - e_6$	$f_{26}f_{45}$	g_1e_3	$f_{24}f_{56}$	$f_{46}f_{25}$	g_3e_1
$\ell - e_6$	e_2f_{26}	e_4f_{46}	e_3f_{36}	e_5f_{56}	$f_{16}e_1$
$2\ell - e_1 - e_2 - e_3 - e_4$	$f_{23}f_{14}$	g_6e_5	$f_{34}f_{12}$	$f_{24}f_{13}$	e_6g_5
$2\ell - e_3 - e_4 - e_5 - e_6$	e_2g_1	$f_{45}f_{36}$	$f_{46}f_{35}$	$f_{34}f_{56}$	g_2e_1
$3\ell - 2e_1 - e_2 - e_3 - e_4 - e_5 - e_6$	g_3f_{13}	g_6f_{16}	g_4f_{14}	g_2f_{12}	g_5f_{15}

T2				
$3\ell - e_1 - e_2 - 2e_3 - e_5 - e_6$	$f_{36}f_{13}f_{25}$	$f_{23}f_{36}f_{15}$	$g_4f_{13}e_1$	$f_{35}g_5e_4$
$4\ell - e_1 - 2e_2 - 2e_3 - e_4 - 2e_5 - e_6$	$g_1f_{13}f_{25}$	$f_{23}g_1f_{25}$	$g_6g_4e_1$	$f_{35}f_{25}g_5$
$3\ell - e_1 - e_2 - 2e_3 - e_4 - e_5$	$f_{34}f_{13}f_{25}$	$f_{23}f_{34}f_{25}$	$g_6f_{13}e_1$	$e_6f_{35}g_5$
$2\ell - e_1 - e_2 - e_3$	$e_5f_{13}f_{25}$	$f_{23}e_5f_{15}$	$f_{12}f_{13}e_1$	$e_6e_4g_5$
$3\ell - e_1 - 2e_2 - e_3 - e_4 - e_5$	$f_{24}f_{13}f_{25}$	$f_{23}f_{24}f_{15}$	$g_6f_{12}e_1$	$e_6f_{25}g_5$
$3\ell - e_1 - 2e_2 - e_3 - e_5 - e_6$	$f_{26}f_{13}f_{25}$	$f_{23}f_{26}f_{15}$	$g_4f_{12}e_1$	$e_4f_{25}g_5$
$3\ell - e_1 - e_2 - e_3 - e_5 - 2e_6$	$f_{16}f_{36}f_{25}$	$g_4f_{16}e_1$	$e_4f_{56}g_5$	$f_{26}f_{36}f_{15}$
$4\ell - e_1 - e_2 - 2e_3 - e_4 - 2e_5 - 2e_6$	$g_2f_{36}f_{25}$	$g_2g_4e_1$	$f_{35}f_{56}g_5$	$g_1f_{36}f_{15}$
$3\ell - e_1 - 2e_2 - e_4 - e_5 - e_6$	$f_{12}f_{46}f_{25}$	$g_3f_{12}e_1$	$e_3f_{25}g_5$	$f_{26}f_{24}f_{15}$
$3\ell - e_1 - e_2 - 2e_4 - e_5 - e_6$	$f_{14}f_{46}f_{25}$	$g_3f_{14}e_1$	$f_{45}e_3g_5$	$f_{24}f_{46}f_{15}$
$3\ell - e_1 - e_2 - e_4 - e_5 - 2e_6$	$f_{16}f_{46}f_{25}$	$g_3f_{16}e_1$	$e_3f_{56}g_5$	$f_{26}f_{46}f_{15}$
$4\ell - e_1 - 2e_2 - e_3 - e_4 - 2e_5 - 2e_6$	$g_4f_{46}f_{25}$	$g_3g_4e_1$	$f_{56}f_{25}g_5$	$f_{26}g_1f_{15}$
$4\ell - e_1 - 2e_2 - e_3 - 2e_4 - 2e_5 - e_6$	$g_6f_{46}f_{25}$	$g_3g_6e_1$	$f_{45}f_{25}g_5$	$f_{24}g_1f_{15}$
$4\ell - e_1 - e_2 - e_3 - 2e_4 - 2e_5 - 2e_6$	$g_2f_{46}f_{25}$	$g_3g_2e_1$	$f_{45}f_{56}g_5$	$g_1f_{46}f_{15}$
$3\ell - e_1 - e_2 - e_3 - 2e_4 - e_5$	$f_{34}f_{14}f_{25}$	$g_6f_{14}e_1$	$e_6f_{45}g_5$	$f_{24}f_{34}f_{15}$
$2\ell - e_1 - e_2 - e_4$	$e_5f_{14}f_{25}$	$f_{14}f_{12}e_1$	$e_6e_3g_5$	$f_{24}e_5f_{15}$
$2\ell - e_1 - e_2 - e_6$	$e_5f_{16}f_{25}$	$f_{16}f_{12}e_1$	$e_4e_3f_5$	$f_{26}e_5f_{15}$
$4\ell - e_1 - e_2 - 2e_3 - 2e_4 - 2e_5 - e_6$	$g_2f_{34}f_{25}$	$g_2g_6e_1$	$f_{45}f_{35}g_5$	$f_{34}g_1f_{15}$
$3\ell - e_1 - e_3 - e_4 - e_5 - 2e_6$	$f_{46}f_{13}f_{56}$	$e_2f_{56}g_5$	$g_2f_{16}e_1$	$f_{46}f_{36}f_{15}$
$4\ell - e_1 - e_2 - 2e_3 - e_4 - 2e_5 - 2e_6$	$g_1f_{13}f_{56}$	$g_2g_4e_1$	$f_{35}f_{56}g_5$	$g_1f_{36}f_{15}$
$3\ell - e_1 - 2e_3 - e_4 - e_5 - e_6$	$f_{34}f_{13}f_{56}$	$e_2f_{35}g_5$	$g_2f_{13}e_1$	$f_{34}f_{36}f_{15}$
$2\ell - e_1 - e_3 - e_5$	$e_5f_{13}f_{56}$	$e_2e_4g_5$	$f_{16}f_{13}e_1$	$e_5f_{36}f_{15}$
$3\ell - e_1 - e_2 - e_3 - e_5 - 2e_6$	$f_{26}f_{13}f_{56}$	$g_4f_{16}e_1$	$e_4f_{56}g_5$	$f_{26}f_{36}f_{15}$
$3\ell - e_1 - e_3 - 2e_4 - e_5 - e_6$	$f_{34}f_{14}f_{56}$	$e_2f_{45}g_5$	$g_2f_{14}e_1$	$f_{34}f_{46}f_{15}$
$2\ell - e_1 - e_4 - e_6$	$e_5f_{14}f_{56}$	$e_2e_3g_5$	$f_{16}f_{14}e_1$	$e_5f_{46}f_{15}$
$2\ell - e_1 - e_4 - e_6$	$f_{46}e_3f_{13}$	$e_2e_3g_5$	$f_{16}f_{14}e_1$	$e_5f_{46}f_{15}$
$2\ell - e_1 - e_3 - e_4$	$f_{34}e_3f_{13}$	$e_2e_6g_5$	$f_{14}f_{13}e_1$	$e_5f_{34}f_{15}$
$2\ell - e_1 - e_4 - e_5$	$f_{45}e_3f_{13}$	$e_6f_{14}f_{56}$	$f_{34}e_3f_{15}$	$f_{14}e_1f_{15}$
$2\ell - e_1 - e_2 - e_4$	$f_{24}e_3f_{13}$	$f_{14}f_{12}e_1$	$e_6e_3g_5$	$f_{24}e_5f_{15}$
$2\ell - e_1 - e_2 - e_6$	$f_{26}e_3f_{13}$	$f_{16}f_{12}e_1$	$e_4e_3g_5$	$f_{26}e_5f_{15}$
$3\ell - e_1 - e_3 - 2e_4 - e_5 - e_6$	$f_{45}f_{46}f_{13}$	$e_2f_{45}g_5$	$g_2f_{14}e_1$	$f_{34}f_{46}f_{15}$
$4\ell - e_1 - e_2 - 2e_3 - 2e_4 - 2e_5 - e_6$	$f_{45}g_1f_{13}$	$g_2g_6e_1$	$f_{45}f_{35}g_5$	$f_{34}g_1f_{15}$
$3\ell - e_1 - e_2 - e_3 - 2e_4 - e_5$	$f_{24}f_{45}f_{13}$	$g_6f_{14}e_1$	$e_6f_{45}g_5$	$f_{24}f_{34}f_{15}$
$5\ell - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5 - 2e_6$	$g_2f_{12}g_1$	$g_6g_4f_{46}$	$g_2f_{25}g_5$	$g_1g_5f_{15}$

and the cubic generator of degree $-K$ given by the first monomial in the table below.

$3\ell - e_1 - e_2 - e_3 - e_4 - e_5 - e_6$	$f_{46}f_{13}f_{25}$	$e_2f_{25}g_5$	$f_{23}f_{46}f_{15}$	$g_2f_{12}e_1$	$f_{26}f_{34}f_{15}$

Lemma 50. *For every Del Pezzo surface X_6, the monomials*

$$f_{34}f_{16}f_{25} \quad f_{46}f_{13}f_{25} \quad f_{23}f_{46}f_{15} \quad g_2f_{12}e_1 \quad f_{26}f_{34}f_{15}$$

are linearly dependent modulo Q_6.

Proof. The ideal Q_6 contains elements

$$\begin{aligned} p_1 &= f_{12}\big(a_1f_{46}f_{35} + a_2f_{34}f_{56} + a_3g_2e_1\big) \\ p_2 &= f_{34}\big(b_1f_{12}f_{56} + b_2f_{16}f_{25} + b_3f_{26}f_{15}\big) \\ p_3 &= f_{46}\big(c_1f_{12}f_{35} + c_2f_{13}f_{25} + c_3f_{23}f_{15}\big) \end{aligned}$$

with a_i, b_i, c_i nonzero constants. Thus $p_1 - \frac{a_1}{c_1}p_3$ is a nonzero polynomial of the form

$$d_1f_{34}f_{12}f_{56} + d_2f_{46}f_{13}f_{25} + d_3f_{23}f_{46}f_{15} + d_4g_2f_{12}e_1$$

with $d_1 \neq 0$ since otherwise, evaluation at a point on the strict transform of the line through points 4 and 6 and in no other exceptional curve would show that all coefficients are zero. As a result $p_1 - \frac{a_1}{c_1}p_3 - \frac{b_1}{d_1}p_2 \in Q_6$ is the required linear dependency relation. $\square$

Lemma 51. *If X_6 is a Del Pezzo surface without Eckart points, the ideal Q_6 contains an element of the form*

$$a_1f_{46}f_{13}f_{25} + a_2e_2f_{25}g_5 + a_3f_{23}f_{46}f_{15} + a_4g_2f_{12}e_1 + a_5f_{26}f_{34}f_{15}$$

where the a_i are constants and $a_1 \neq 0$.

Proof. By Lemma 50 the ideal Q_6 contains a nonzero element of the form

$$s = b_1f_{34}f_{16}f_{25} + b_2f_{46}f_{13}f_{25} + b_3f_{23}f_{46}f_{15} + b_4g_2f_{12}e_1 + b_5f_{26}f_{34}f_{15}$$

We show that b_1 is nonzero if the surface has no Eckart points. Otherwise evaluation at the intersection points $f_{46} \cap f_{15}$ and $f_{46} \cap f_{12}$ shows that $b_4 = b_5 = 0$ since no triple of exceptional curves has a common point. This forces $b_2 = b_3 = 0$ yielding a contradiction.
Now Q_6 also contains a relation of the form

$$r = f_{25}\big(c_1f_{34}f_{16} + c_2f_{46}f_{13} + c_3e_2g_5\big)$$

so $r - \frac{c_1}{b_1}s$ is a nonzero element of Q_6 of the desired form and the reasoning of the first paragraph (on the pairs of curves f_{15}, g_5 and f_{15}, e_1) shows that if X_6 has no Eckart points then $a_1 \neq 0$. $\square$

The Table T4 contains the weight vectors w which lead to all quadratic initial ideals of C_5 (up to the action of the Weyl Group $W_5 = D_5$).

$T4$															
f_{12}	f_{13}	f_{23}	f_{14}	f_{24}	f_{34}	f_{15}	f_{25}	f_{35}	f_{45}	e_1	e_2	e_3	e_4	e_5	g
25	21	11	19	13	9	13	9	1	11	10	10	10	10	10	10
18	15	12	14	10	9	12	10	11	5	10	10	10	10	10	10
25	19	15	17	11	9	15	13	17	3	10	10	10	10	10	10
17	15	13	14	11	10	12	11	12	5	10	10	10	10	10	10
19	12	15	11	13	10	10	14	8	5	10	10	10	10	10	10
20	12	15	9	11	4	11	16	10	10	10	10	10	10	10	10
39	31	23	25	21	19	27	21	17	9	20	20	20	20	20	20
17	15	10	14	11	9	12	10	6	12	10	10	10	10	10	10
27	19	17	17	11	9	17	13	19	3	10	10	10	10	10	10
18	17	13	15	12	11	11	10	8	11	10	10	10	10	10	10
18	16	13	14	12	11	11	10	7	11	10	10	10	10	10	10
20	14	16	10	11	6	12	15	11	11	10	10	10	10	10	10
18	16	12	14	11	10	11	9	6	10	10	10	10	10	10	10
25	21	13	19	15	11	13	11	3	13	10	10	10	10	10	10
18	15	13	14	11	10	12	11	12	6	10	10	10	10	10	10
20	13	16	9	11	5	11	15	10	10	10	10	10	10	10	10
20	13	15	10	11	5	12	16	11	11	10	10	10	10	10	10
21	27	15	17	11	13	13	13	11	1	10	10	10	10	10	10

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